Physical layer models and techniques for software radio

ASYNCHRONOUS DIRECT SEQUENCE SPREAD SPECTRUM

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References

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Introduction

Direct Sequence Spread Spectrum system can be used as a modulation technique to allow two or more users to share the same band provided that different spreading codes are assigned to different users.

Each user generate a cross interference on other users called

- **Multi User Interference** (MUI)

whose power is related to Process Gain ***N*.** By choosing particular families of spreading codes, MUI effects can be reduced.

Code Division Multiple Access (CDMA) is the multiple access technique based on **Spread Spectrum.**

CDMA allows users to jointly transmit, over the same band and with no predefined temporal limitations (**asynchronous** transmission).

Multi User DS-CDMA

In <u>Multi-user DS-CDMA</u> each transmitter is identified by a **<u>PN sequence.</u>**

The simplest ("conventional") single-user receiver detects transmitted information by using **a matched filter** cross-correlating the specific user PN sequence with the received signal.

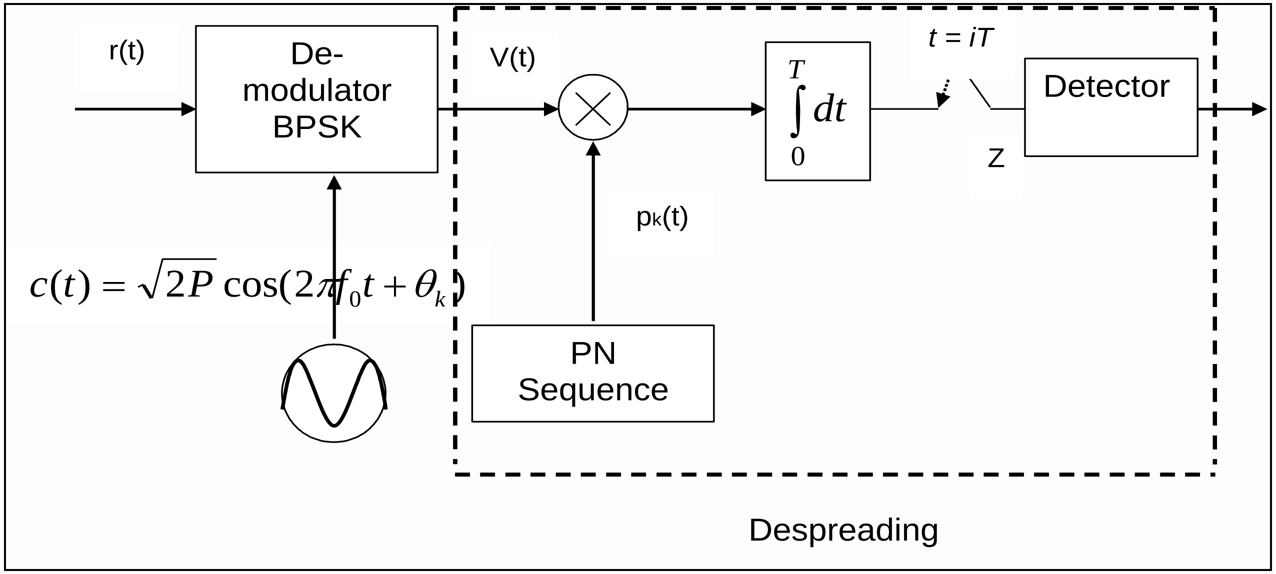
For this receiver, DS Spread signals of other users, simultaneously transmitting, are considered as **Multi User Interference, MUI**.

MUI generally cannot be represented as a Gaussian noise.

In DS-CDMA a bank of K receivers as the ones described in next slide should be present at the receiver to allow each user information to be detected.

Multi User DS-CDMA

The received signal of user k, after being matched with PN code of user k, $p_k(t)$, and sampled at time iT, i being an integer, can be written as the contribution Z_i of three components:



$$Z_i = \sqrt{\frac{P}{2}} T b_{k,i} + \eta + I = \sqrt{\frac{P}{2}} T b_{k,i} + n_g$$

- First Term is proportional to i-th received bit of user k.
- η is the AWGN, I is the MUI
- $n_g = \eta + I$ is global noise

Perfomances –
MUI as AWGN hypothesis

In real systems **the number K of users can be high**; consequently, the overall interference (MUI) generated by K-1 users on kth user can in such cases be considered as **Gaussian, thanks to central limit theorem**.

In addition, one can as a first approximation consider PN broad band noise generated by interfering users as **White noise** (flat infinite spectrum)

Such hypothesis can simplify Bit error rate (BER) computation.

In particular, considering:

- (*K-1*) DS-SS interfering users (K large),
- K-1 users interference power at the receiver in baseband *B* being equal to *(k-1)P,* where *P* is the received power, here considered equal for all interfering users.

Then MUI **spectral density** can be written as:

$$\frac{I_0}{2} = \frac{(K-1)P}{2B}$$

Performances – MUI as AWGN hypothesis

At the detector input, after sampling matched filter output, the **power** of overall noise (including both MUI and AWGN, considered here as independent random variables) can be written as the sum of the power of AWGN and MUI noise, i.e.:

$$N_{TOT} = \left(N_0/2\right)2B + \left(K-1\right)P = N_0 B + (K-1)P$$

So if MUI is assumed to be a AWGN interference, the **Signal to Noise Ratio** at at the detector can be written as the ratio between each i-th bit energy and the sum of MUI and AWGN power:

$$\boxed{SNR_{out} = \frac{E_b}{N_0 + I_0} = \frac{E_b}{N_0 + (K-1)P/B}}$$

Performances– MUI as AWGN hypothesis

As a BPSK modulator is here used BER is $P_{E,BPSK} = Q\left(\sqrt{2SNR_{out}}\right)$

So , if we approximate MUI as a White Gaussian noise than, recalling that modulated bandwidth is $B \cong \frac{2}{T_c}$, $E_b = PT$ and writing process gain as $N = \frac{T}{T_c}$ than :

$$P_{E,BPSK} \approx Q\left(\left[\frac{K-1}{4N} + \frac{1}{2E_b/N_0}\right]^{-1/2}\right)$$

where the Gaussian error function is $Q(x) \hat{=} \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$

As a limit case, when K=1 (single user) also from the previous approximated formula one could conclude DS-SS having the **same performances** of a narrow band BPSK modulation.

However, when K is small Gaussian hypothesis over MUI does not hold. Moreover, MUI is not white noise as generated by PN sequences.

BER Evaluation – MUI as Gaussian (not White) noise

In general MUI cannot be modeled as white noise.

In real case spectral density of MUI is NOT flat, but depends on PN sequences spectral density.

If white noise hypothesis is relaxed but **MUI Gaussianity is still assumed** (e.g., still valid with K large), first and second order statistics of MUI can be used to describe expected performances.

Let global interference $n_g = \eta + I$ be Gaussian.

This holds if both noise components after matched filter, *η* and *I are assumed to be* Gaussian distributed, *where η represents AWGN and I is MUI.*

So a second more precise approximation of performances of conventional detector can be obtained by assuming just that MUI is Gaussian (and not White)

Perfomances – Gaussian MUI hypothesis

It is easy to show that

$$E\{I\} = E\{n_g\} = 0$$

while (due to independence of MUI and AWGN).

$$\mathrm{var}(n_g) = \mathrm{var}(\eta) + \mathrm{var}(I)$$

η is the AWGN sampled component at the output of the matched filter of user k when *n(t)* is the input, i.e.:

$$\eta = \int_0^T n(t)\, p_1(t) \cos(\omega_c t)\, dt$$

whose variance is $N_0 T/4$

I, is the multi user interference component generated by K-1 users at the output of the matched filter of user 1,. For notation simplicity index of ith bit is omitted and a generic bit is assumed. I it can be written as:

$$I \hat{=} \sqrt{\frac{P}{2}} \sum_{k=2}^{K} Z_{k,1} = \sqrt{\frac{P}{2}} \sum_{\mathbf{k=2}}^{\mathbf{K}} \left[\int_0^T b_k(t-\tau_k)\, p_k(t-\tau_k)\, p_1(t) dt \right] \cos(\phi_k)$$

where ϕ_k is the phase delay and τ_k is the time delay for user *k*

Gaussian MUI hypothesis

In fact, by assuming the source of user kth emits equally probable binary symbols, than due to symmetrical shape of Gaussian variable Z around its zero mean, it can be written that:

$$\Pr\left\{{}^{Z<0}\!/_{b_k(1)=1}\right\}=\Pr\left\{{}^{Z\geq 0}\!/_{b_k(1)=-1}\right\}$$

So error probability reduces to the following expression:

$$P_{err}=\frac{1}{2}\Pr\left\{{}^{Z\geq 0}\!/_{b_k(1)=-1}\right\}+\frac{1}{2}\Pr\left\{{}^{Z<0}\!/_{b_k(1)=1}\right\}=\Pr\left\{{}^{Z\geq 0}\!/_{b_k(1)=-1}\right\}$$

By observing that $Z=n_g+\sqrt{\frac{P}{2}}T$ and due to symmetry around zero of global noise, than $\Pr\left\{{}^{Z\geq 0}\!/_{b_k(1)=-1}\right\}=\Pr\left\{{}^{n_g\geq\sqrt{\frac{P}{2}}T}\!/_{b_k(1)=-1}\right\}$

So $P_{err}=\int_{\sqrt{\frac{P}{2}}T}^{+\infty}G\left(0,var(n_g)\right)\mathrm{d}n_g$

where $G\left(0,var(n_g)\right)$ is zero-mean Gaussian *pdf* of n_g

Perfomances –
Gaussian MUI hypothesis

Error probability can be also written as:

$$P_{err} = Q\left(\sqrt{2SNR_{out}}\right) = \mathrm{Q}\left(\sqrt{\frac{\frac{P}{2}T^2}{var(n_g)}}\right) = \mathrm{Q}\left(\sqrt{\frac{\frac{P}{2}T^2}{\frac{N_0}{T}+var(I)}}\right)=\mathrm{Q}\left(\sqrt{\frac{1}{\frac{N_0}{2E_b}+\frac{var(I)}{\frac{P}{2}T^2}}}\right)$$

where the term to be computed reduces to variance of MUI, $var(I)$, and N_0 is AWGN power.

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By comparing the SNR with the one obtained in MUI approximation assuming MUI as white noise, it comes out that

$$\mathrm{Q}\left(\sqrt{\frac{1}{\frac{N_0}{2E_b}+\frac{var(I)}{\frac{P}{2}T^2}}}\right)=\mathrm{Q}\left(\left[E\{\Psi^2\}+\frac{1}{\frac{2E_b}{N_0}}\right]^{-\frac{1}{2}}\right)$$ where

$\Psi = \frac{I}{\sqrt{\frac{P}{2}T}}$ is the normalized multi-user interference and $E\{\Psi^2\}$

substitutes the term $\frac{K-1}{4N}$ that was used in the previous white Gaussian MUI model

MUI Variance

Variance of *I*, *var(I),* or equivalently $\mathrm{var}(\Psi)$, have to be computed to obtain P_{err}. Such variances depend on PN codes of users, statistics of source data, modulator oscillator phase delays and delay spread at the receiver.

$$E(\Psi)=0$$

$$E(\Psi^2)=\sum_{k=2}^{K}E_{b,\tau,\phi}\left(\left\{[b_k(0)\rho_{k,1}(\tau_k)+b_k(1)\hat{\rho}_{k,1}(\tau_k)]\cos(\phi_k)\right\}^2\right)$$

where $\rho_{k,1}(\tau)\triangleq\int_o^{\tau}p_k(t-\tau)p_1(\tau)d\tau$ and $\hat{\rho}_{k,1}(\tau)\triangleq\int_{\tau}^{T}p_k(t-\tau)p_1(\tau)d\tau$

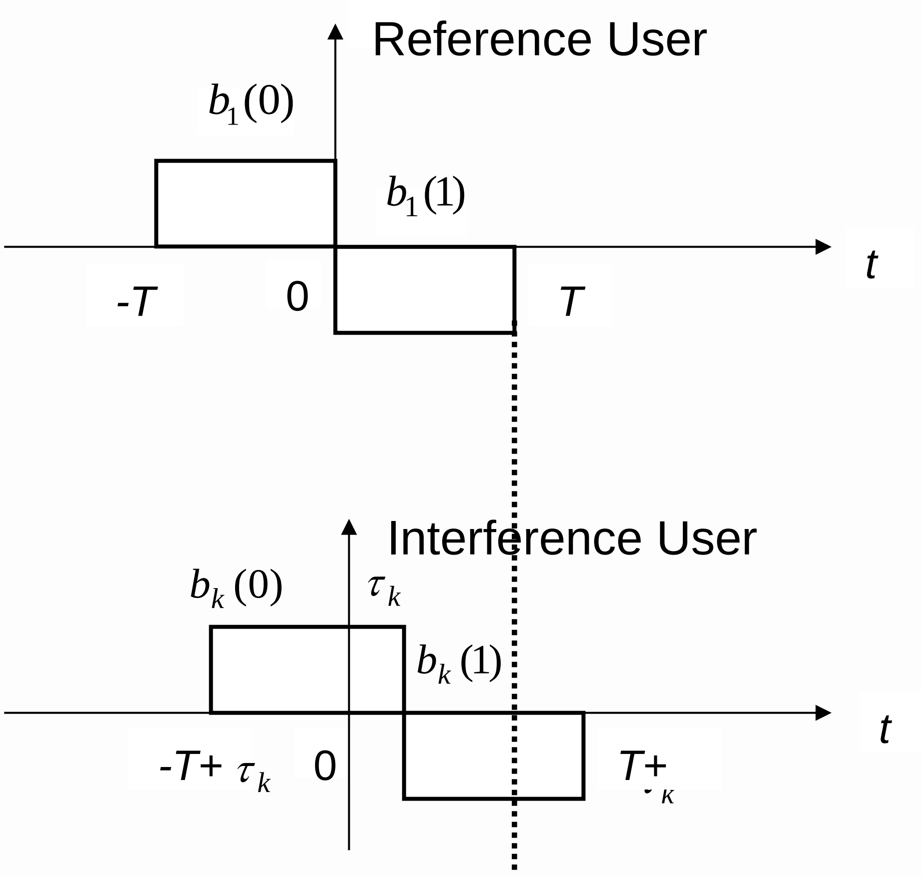
Note: time delay and phase delay are uniformly distributed variables in [0,T) and [0,2π) and the transmitted symbols are equally probable.

The two integrals evaluate the cross-similarity of codes of users 1 and interfering user k over two consecutive bits transmitted by interfering user

Example

Baseband interference of user k on user 1 is described when user k is asynchronous wrt user 1 with delay τ_k

The matched filter of user 1 is supposed to be time synchronized at the receiver



MUI Variance

Cross similarity terms among PN codes used to compute MUI variance can be written by considering the a-periodic discrete cross-correlation between PN sequences of reference user 1 and PN sequence of user *K.*

$$\chi_{k,1}(l) = \begin{cases} \sum_{j=0}^{N-l-1} p_k(j)\, p_1(\mathrm{j}+\mathrm{l}) & 0 \le l \le N-1 \\ \sum_{j=0}^{N+l-1} p_k(j-l)\, p_1(\mathrm{l}) & 1-N \le l \le 0 \end{cases}$$

The continuous integrals from slide 10 can be computed using above results as:

$$\hat{\rho}_{k,1}(\tau_k) = \chi_{k,1}(l_k)T_c + \{\chi_{k,1}(l_k+1) - \chi_{k,1}(l_k)\}(\tau_k - l_k T_c)$$

$$\rho_{k,1}(\tau_k) = \chi_{k,1}(l_k - N)T_c + \{\chi_{k,1}(l_k - N + 1) - \chi_{k,1}(l_k - N)\}(\tau_k - l_k T_c)$$

for l_k such as $l_k T_c \le \tau \le (l_k + 1)T_c$

MUI Variance

By considering independence between time delays, phase delays and source data, variance of normalized MUI can be recomputed as:

$$E(\Psi^2) = \sum_{k=2}^{K} E_\tau \left\{\rho_{k,1}^2(\tau_k) + \hat{\rho}_{k,1}^2(\tau_k)\right\} E_\phi \left\{\cos^2(\phi_k)\right\}$$

where $E_\phi \left\{\cos^2(\phi_k)\right\} = \frac{1}{2\pi}\int_0^{2\pi} (\cos^2 \phi) d\phi = \frac{1}{2}$ and

$$E_\tau \left\{\rho_{k,1}^2(\tau_k) + \hat{\rho}_{k,1}^2(\tau_k)\right\} = \frac{1}{T}\int_0^{T} \left\{\rho_{k,1}^2(\tau) + \hat{\rho}_{k,1}^2(\tau)\right\} d\tau$$

This latter integral can be divided in a summation of all integrals in the interval

$[lT_c, (l+1)T_c]$ where $0 \le l \le N-1$.

$$E_\tau \left\{\rho_{k,1}^2(\tau_k) + \hat{\rho}_{k,1}^2(\tau_k)\right\} = \frac{1}{T}\sum_{l=0}^{N-1} \int_{lT_c}^{(l+1)T_c} \left\{\rho_{k,1}^2(\tau) + \hat{\rho}_{k,1}^2(\tau)\right\} d\tau$$

MUI Variance

By substituting

- the integral of the sum with the sum of integrals
- $\left\{\rho_{k,1}^2(\tau) + \hat{\rho}_{k,1}^2(\tau)\right\}$ with obtained expressions, the variance can be written as:

$$E\left(\Psi^2\right) = \frac{1}{6N^3}\sum_{k=2}^{K}\sum_{l=0}^{N-1}\left[f_v(a_{k,l}, b_{k,l}, \hat{a}_{k,l}, \hat{b}_{k,l})\right]$$

where

$$a_{k,l} \triangleq \chi_{k,1}(l-N) \qquad \hat{a}_{k,l} \triangleq \chi_{k,1}(l)$$

$$b_{k,l} \triangleq \chi_{k,1}(l-N+1) \qquad \hat{b}_{k,l} \triangleq \chi_{k,1}(l+1)$$

$$f_v(x,y,z,w) \triangleq x^2 + y^2 + z^2 + w^2 + xy + zw$$

Conclusion MUI Variance

The last formula allows one to conclude two former aspects that emerge from both MUI models:

1- The higher the processing gain N, the lower MUI variance, i.e. by increasing by Spreading the bandwidth, MUI can be reduced.

2 - The higher the number of users K the higher MUI

In addition the latter MUI model allows one to conclude that:

3- Appropriate choice of PN sequences can reduce MUI. A low cross correlation among codes will reduce MUI.

Conclusions: BER Evaluation

For an asynchronous DS/CDMA system, BER expression can be written (partially reported in slide 14)[5] as:

$$E\left(\Psi^2\right) \hat{=} \frac{1}{6N^3}\sum_{k=2}^{K}\sum_{l=0}^{N-1}\left[f_v(a_{k,l}, b_{k,l}, \hat{a}_{k,l}, \hat{b}_{k,l})\right] \approx \frac{K-1}{3N}$$

It leads to:

$$SNR_{out} \hat{=} \sqrt{\frac{\mathrm{E}}{\mathrm{var}(I)+\mathrm{var}(\eta)}} = \left(E\left(\Psi^2\right)+\frac{1}{2}(SNR)^{-1}\right)^{-1/2} \approx \left\{\frac{K-1}{3N}+\frac{1}{2SNR}\right\}^{-1/2}$$

•If **stochastic PN sequences** are considered: $E\left(\Psi^2\right)=\frac{K-1}{3N}$

This formulation is wrong for “few users” whereas can be used for large number of users. It is useful for a simple evaluation of DS/CDMA system performances

$$P_E \approx Q\left[\left\{\frac{K-1}{3N}+\frac{1}{2SNR}\right\}^{-1/2}\right]$$

Conclusions: BER Evaluation (2/3)

From P_E expression can be derived an evaluation of CDMA system capacity, in terms of number simultaneous users served with a certain Quality of Service (QoS)

For high values of x:

$$Q(x) \approx \frac{\exp(-x^2/2)}{\sqrt{2\pi}\,x}$$

Considering admissible $P_E \approx 10^{-3}$ (sufficient for vocal applications)

$$Q(3.11) \approx 10^{-3} \longrightarrow K = 3N\left(\frac{1}{(3.11)^2} - \frac{1}{2E_b/N_0}\right) + 1$$

Considering the right side of equation as upper bound:

$$K < 3N\left(\frac{1}{(3.11)^2} + \frac{1}{2E_b/N_0}\right) + 1$$

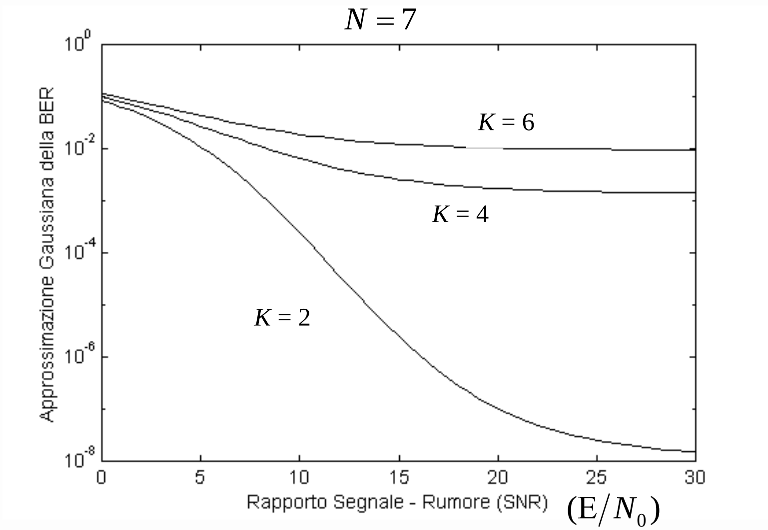
Conclusions: BER Evaluation (3/3)

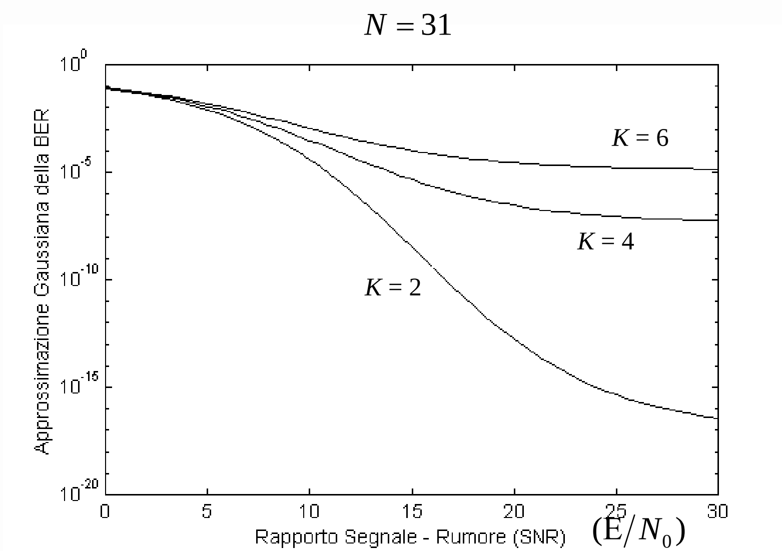
For **high values of signal-to-noise ratio** an approximation is possible:

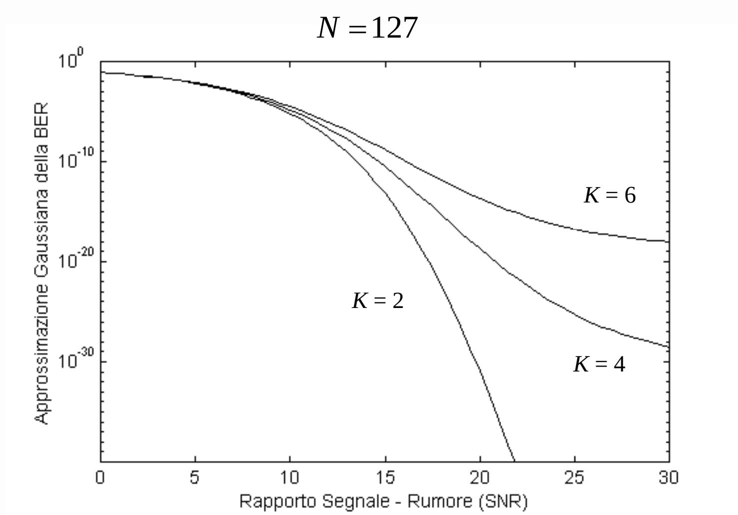
$$K \approx \frac{N}{3}$$

As a simple rule of thumb for a DS/CDMA system, system capacity (number of asynchronous users that can maximally be served) is *N/3*, where *N* is the process gain, with a probability error lower than 10^{-3}.

Example: numerical results







- BER Gaussian evaluation for DS/CDMA systems
- BPSK modulation
- Gold sequences
- *K* = number of users

Comments

BER Gaussian evaluation is only an *approximation* of real BER.

For SNR < 10 dB, Gaussian noise is predominant and BER is barely influenced by new users.

For very high SNRs, MUI is predominant and the higher the number of users, the lower are performances, if process gain is low.

Increasing SNR over a certain threshold, BER saturates: this is the bottle-neck given by MUI presence.

To increase performances, a higher process gain is needed; this fact involves an expansion of transmission band, at the equal bit-rate.

Multi-user Detectors

The conventional single user detector can act as an optimal detector only when K=1 i.e. when a single user is present on the CDMA system.

MUI always limits it performances at higher probability of error when multiple users are present. Despite codes could be in principle chosen to minimize effects of MUI, it can be seen in practice that choosing family of codes that fully eliminate MUI in asynchronous transmission is not possible.

A solution is to explore whether more complex detectors can be realized at the receiver that could filter out MUI, eventually being provided of additional knowledge.

Optimal Multi user DS-CDMA Receiver

Let us define the optimal receiver in case the transmitted signal s(t) is composed by K asynchronously superimposed DS-SS signals from K users; the received r(t) signal, can then be expressed as:

$$r(t) = s(t) + n(t)$$

where *n(t)* is noise with flat spectral density $\frac{N_0}{2}$.

An optimal receiver, by definition, has to jointly select a bit sequence of all N bits transmitted by K users that optimize a detection functional:

$$\left\{ \hat{b}_k(n), 1 \le n \le N, 1 \le k \le K \right\}$$

As a functional here we can choose the most a posteriori probable (MAP) criterion, i.e. the most probable sequence of bits conditioned to received signal *r(t)* observed during a temporal period $0 \le t \le NT+2T$, *i.e.:*

$$\left\{ \hat{b}_k(n) \right\} = \arg \max_{b_k(n)} P\left(b_k(n) \big/ r(t), t \in (0, NT + 2T) \right)$$

Optimal Multi user DS-CDMA Receiver

It is well known that, in case of equally probable K user sources, i.e. if

$$Pr(b_k(n) = +1) = Pr(b_k(n) = -1)$$

than MAP criterion becomes equivalent to Maximum Likelihood.

$$\underset{\{b_k(n)\}}{\operatorname{argmax}} Pr\left({}^{\{b_k(n)\}}\!/_{r(t)} \right) = \underset{\{b_k(n)\}}{\operatorname{argmax}} Pr\left({}^{r(t)}\!/_{\{b_k(n)\}} \right)$$

Moreover by taking into account the fact that noise is Gaussian, than:

$$\underset{\{b_k(n)\}}{\operatorname{argmax}} Pr\left({}^{r(t)}\!/_{\{b_k(n)\}} \right) = \underset{\{b_k(n)\}}{\operatorname{argmax}} Pr\left({}^{n(t)=r(t)-s(t)}\!/_{\{b_k(n)\}} \right)$$

and the most probable value is when the variance

$$\underset{\{b_k(n)\}}{\operatorname{argm}in}\, E\left\{ {}^{n^2(t)}\!/_{\{b_k(n)\}} \right\}$$

$$= \underset{\{b_k(n)\}}{\operatorname{argm}in}\, E\left\{ {}^{\left(r(t)-s(t)\right)^2}\!/_{\{b_k(n)\}} \right\} = \underset{\{b_k(n)\}}{\operatorname{argm}in}\, \Lambda(\{b_k(n)\})$$

For all time instants $t \in \{0, NT + 2T\}$

$\Lambda(\{b_k(n)\}) = \Lambda(\boldsymbol{b})$ is the likelihood function

Optimal Multi user DS-CDMA Receiver

Two consecutive symbols from each user interfere with the reference user signal. Let us assume that receiver knows energies of signals $\{E_k\}$ and their transmission delays $\{\tau_k\}$

If symbols are equally probable, optimal MAP receiver (equivalent to ML Maximum Likelihood) can search for optimal vector **b** by minimizing a Gaussian likelihood function, $\Lambda(\boldsymbol{b})$, where:

$$\Lambda(\mathbf{b}) = \int_0^{NT+2T} \left[r(t) - \sum_{k=1}^{K} \sqrt{E_k} \sum_{i=1}^{N} b_k(i) c_k(t - iT - \tau_k) \right]^2 dt =$$

$$= \int_0^{NT+2T} r^2(t)dt - 2\sum_{k=1}^{K} \sqrt{E_k} \sum_{i=1}^{N} b_k(i) \int_0^{NT+2T} r(t) c_k(t - iT - \tau_k) dt +$$

$$+ \sum_{k=1}^{K} \sum_{l=1}^{K} \sqrt{E_k E_l} \sum_{i=1}^{N} \sum_{j=1}^{N} b_k(i) b_l(j) \int_0^{NT+2T} c_k(t - iT - \tau_k) c_l(t - jT - \tau_l) dt$$

Where ***b*** represents the unknown bits sequences received from *K* users

Optimal Multi user DS-CDMA Receiver

To solve the minimization problem, let us consider different elements of the sum The first integral:

$$\int_0^{NT+2T} r^2(t)dt$$

doesn't depend on transmitted bits $\{b_k(n)\}$, so it can be ignored in the minimization.

Within the second term, the following integral is present:

$$r_k(i) = \int_{iT+\tau_k}^{(i+1)T+\tau_k} r(t)c_k(t - iT - \tau_k)dt \qquad 1 \le i \le N$$

It can be noticed that each term $r_k(i)$ represents the output of matched filter outputs of user k in each signal interval corresponding to i-th bit in the sequence.

$$\{\ r_k(i), 1 \le k \le K, 1 \le i \le N\ \}$$

Optimal Multi user DS-CDMA Receiver

Within the third term the following integral can be found that corresponds to cross correlation among PN codes of users k and l.

$$\int_0^{NT+2T} c_k(t - iT - \tau_k)c_l(t - jT - \tau_l)dt$$

$$= \sum_{i=1}^{N} \int_{iT+\tau_k}^{(i+1)T+\tau_k} c_k(t)c_l(t + iT - jT + \tau_k - \tau_l)dt$$

and by using with j-i=m then

$$\rho_{kl}(mT + \tau) = \int_{iT+\tau_k}^{(i+1)T+\tau_k} c_k(t - \tau)c_l(t)dt$$

$\sum_{i=1}^{N} \int_{iT+\tau_k}^{(i+1)T+\tau_k} c_k(t)c_l(t + iT - jT + \tau_k - \tau_l)dt = \sum_{i=1}^{N} \rho_{kl}(jT - iT - \tau_k + \tau_l) =$
$\sum_{i=1}^{N} \rho_{kl}(mT + \tau_l - \tau_k)$

The term $\rho_{kl}(mT + \tau)$ is different from zero only if m=-1,0, +1

Optimal Multi user DS-CDMA Receiver

A vectorial notation can be used to represent the set quantities before described and to describe $\Lambda(\boldsymbol{b})$ in a more compact way. In particular, letting:

$$\mathbf{r} = \left[\mathbf{r}^t(1)\,\mathbf{r}^t(2) \quad \mathbf{r}^t(N)\right]^t \qquad \mathbf{r}(i) = \left[r_1(i)\; r_2(i) \quad r_K(i)\right]^t$$

$$\mathbf{b} = \left[\mathbf{b}^t(1)\,\mathbf{b}^t(2) \quad \mathbf{b}^t(N)\right]^t \qquad \boldsymbol{b}(i) = \left[\sqrt{\mathcal{E}_1}b_1(i)\; \sqrt{\mathcal{E}_2}b_2(i) \quad \sqrt{\mathcal{E}_K}b_K(i)\right]^t$$

$$\boldsymbol{n} = \left[\boldsymbol{n}^t(1)\,\boldsymbol{n}^t(2) \quad \boldsymbol{n}^t(N)\right]^t \qquad \boldsymbol{n}(i) = \left[n_1(i)\; n_2(i) \quad n_K(i)\right]^t$$

Optimal Multi user DS-CDMA Receiver

$$\mathbf{R}_N = \begin{bmatrix} \mathbf{R}_a^t(0) & \mathbf{R}_a^t(1) & 0 & ..... & ..... & 0 \\ \mathbf{R}_a^t(1) & \mathbf{R}_a^t(0) & \mathbf{R}_a^t(1) & 0 & ..... & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \mathbf{R}_a^t(1) & \mathbf{R}_a^t(0) & \mathbf{R}_a^t(1) \\ 0 & 0 & 0 & 0 & \mathbf{R}_a^t(1) & \mathbf{R}_a^t(0) \end{bmatrix}$$

Where $\boldsymbol{R_N}(\boldsymbol{\tau})$ is a *KN*x*KN* matrix whose elements are N KxK matrixes $\boldsymbol{R_a}(\boldsymbol{m})$
Each element (k,l) of $\boldsymbol{R_a}(\boldsymbol{m})$ is given by cross correlation of PN codes of users k and l computed for two bits whose distance in transmission order was of m elements, i.e. $\rho_{kl}(mT+\tau)$
As $\rho_{kl}(mT+\tau)$=0 when $|m|>1$ then all matrixes $\boldsymbol{R_a}(\boldsymbol{m}) = \mathbf{0}$ i.e. are composed by KxK null elements when $|m|>1$

Optimal Multi user DS-CDMA Receiver

Then one can write:

$$\underset{\boldsymbol{b}}{\operatorname{argmin}}\, \Lambda(\boldsymbol{b}) = \underset{\boldsymbol{b}}{\operatorname{argmin}}\, \left(f_2(\boldsymbol{b}, \boldsymbol{r}) + f_3(\boldsymbol{b}, \boldsymbol{R_N})\right)$$

And the optimal receiver can use a brute force method to compute the optimal solution by substituting all possible configurations of vector **b** to find the minimum.

However, this method is largely impractical

The maximum likelihood (MLSE) optimal detector would have to calculate 2^{NK} correlation measures to select the *K* bit sequences of length *N* which correspond to the best correlation measures.

<u>The computational load of this approach is too high for real-time usage.</u>

Optimal Receiver: Alternative Approach (1/2)

Considering maximization of $\Lambda(\boldsymbol{b})$ like a problem of forward dynamic programming can be possible by using **Viterbi algorithm**.

Viterbi algorithm

A key observation is that each transmitted symbol can be overlapped with no more than $2(K-1)$ symbols of other users

$b_1(i)$	
$b_2(i-1)$	$b_2(i)$
$b_K(i-1)$	$b_K(i)$

When the algorithm uses a finite decision delay (a sufficient number of states), the performance degradation becomes negligible

Optimal Receiver: Alternative Approach (2/2)

The previous consideration points out that there is **<u>not a singular method</u>** to decompose $\Lambda(\mathbf{b})$.

Some versions of Viterbi algorithm for multi-user detection, proposed in the state of the art, are characterized by 2^K states and computational complexity $O(4^K/K)$, which is still very high.

This approach is used for a very small number of users (K<10).

<u>When number of users is very high, sub-optimal receivers are considered.</u>

Suboptimal Multi user DS-CDMA Receiver

The vectorial quantities introduced in MLSE detector can be used to model in a geometrical way the channel. For example, a linear expression can be written to model the fact that each matched filter output of user k is affected by MUI from other users as represented by $\boldsymbol{R_N}$:

$$\boldsymbol{r} = \boldsymbol{R_N b} + \boldsymbol{n}$$

In such an expression, Gaussian noise random vector $\boldsymbol{n}$ is a NK independent components , zero mean variables with variance $N_0T/4$

It can be shown that vector $\boldsymbol{r}$ obtained by a bank of conventional matched filters with codes of K users constitutes a sufficient statistic to allow estimation of transmitted bits

Sub-optimal Receivers: Multi user Conventional Receiver

The multi user version of a singlee user conventional receiver can be considered as a suboptimal detector itself. In fact it:

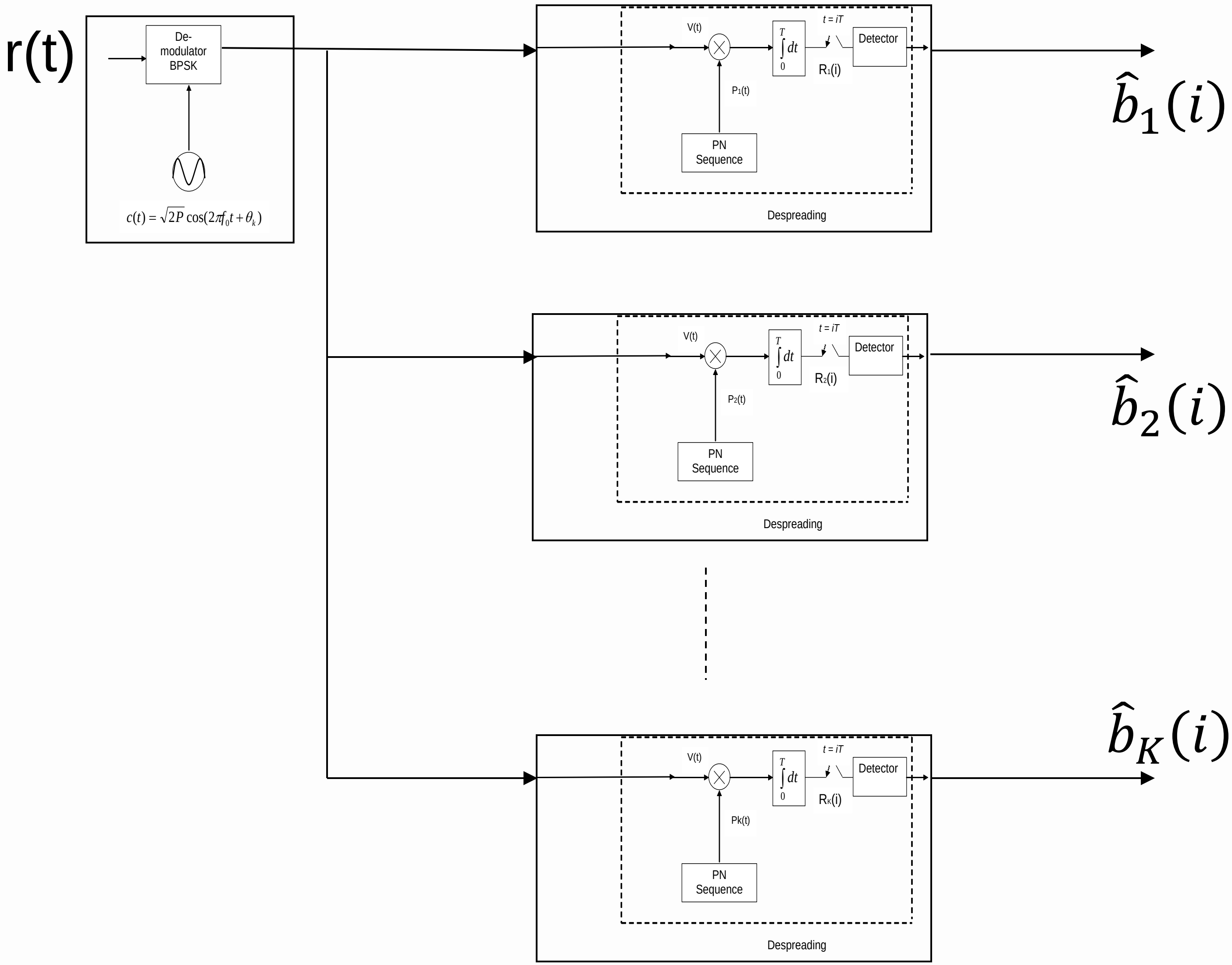
1. Correlates received signal with $p_k(t)$ users' PN sequences.
2. Connects each matched filter output to a detector which implements a decision rule $\hat{b}_k(i) = sgn(r_k(i))$ so obtaining

   $\hat{b}_k(i)$,$i \in [1, N]$, $\mathrm{k} \in [1, K]$.

Conventional receiver for single user is optimal only when noise is **white Gaussian.** In case of multi-users noise is superposition of MUI (not AWGN, in general) and AWGN, So multi-user conventional receiver is suboptimal.

However, conventional multiuser receiver is computationally light, as it requires a complexity qual to the one of K single user conventional receivers.

Sub-optimal Receivers: Multi user Conventional Receiver (1/2)



Sub-optimal Receivers: Multi user Conventional Receiver

The conventional receiver is vulnerable to MUI because it is impossible to design orthogonal sequences, for each couple of users, for any time offset.

The solution can be the definition and choice of sequences with good correlation properties to minimize MUI (e.g. Gold, Kasami PN sequences).

MUI interference can be more critical when other users transmit signals with higher power than the considered signal (**near-far problem**).

Practical solutions require a power control method by the base station by using a separate information channel to transmit power control information monitored by all users.

Other solutions can be other kind of **multi-user detectors**

Sub-optimal Receiver: De-correlating Detector

Decorrelating receiver is a multi-user detector that is based on the observation that the matched filters soft output can be written as: $\boldsymbol{r} = \boldsymbol{R}_N \boldsymbol{b} + \boldsymbol{n}$

Being **n** a NK-multidensional zero mean Gaussian, a multidimensional likelihood function can be minimized as follows:

$$\Lambda_{DC}(\boldsymbol{b}) = (\boldsymbol{r} - \boldsymbol{R}_N \boldsymbol{b})^t \mathbf{Q} (\boldsymbol{r} - \boldsymbol{R}_N \boldsymbol{b})$$

Where Q is white noise covariance.

The solution can be found by searching for

$$\underset{\{b_k(n)\}}{\mathrm{argm}in}\, \Lambda_{DC}(\{b_k(n)\})$$

Sub-optimal Receivers: De-correlating Detector

Maximum likelihood solution $\boldsymbol{b}^{\mathbf{0}}$ is:

$$\boldsymbol{b}^{\mathbf{0}} = \boldsymbol{R}_{\boldsymbol{N}}^{-\mathbf{1}}\boldsymbol{r}$$

The Decorrelating detector ML estimation of ***b*** can so be obtained by using matched filters' parallel bench outputs and passing it to a operator where inverse matrix of coross correlation codes is multiplied .

As $\boldsymbol{r} = \boldsymbol{R}_{\boldsymbol{N}}\boldsymbol{b} + \boldsymbol{n}$ than

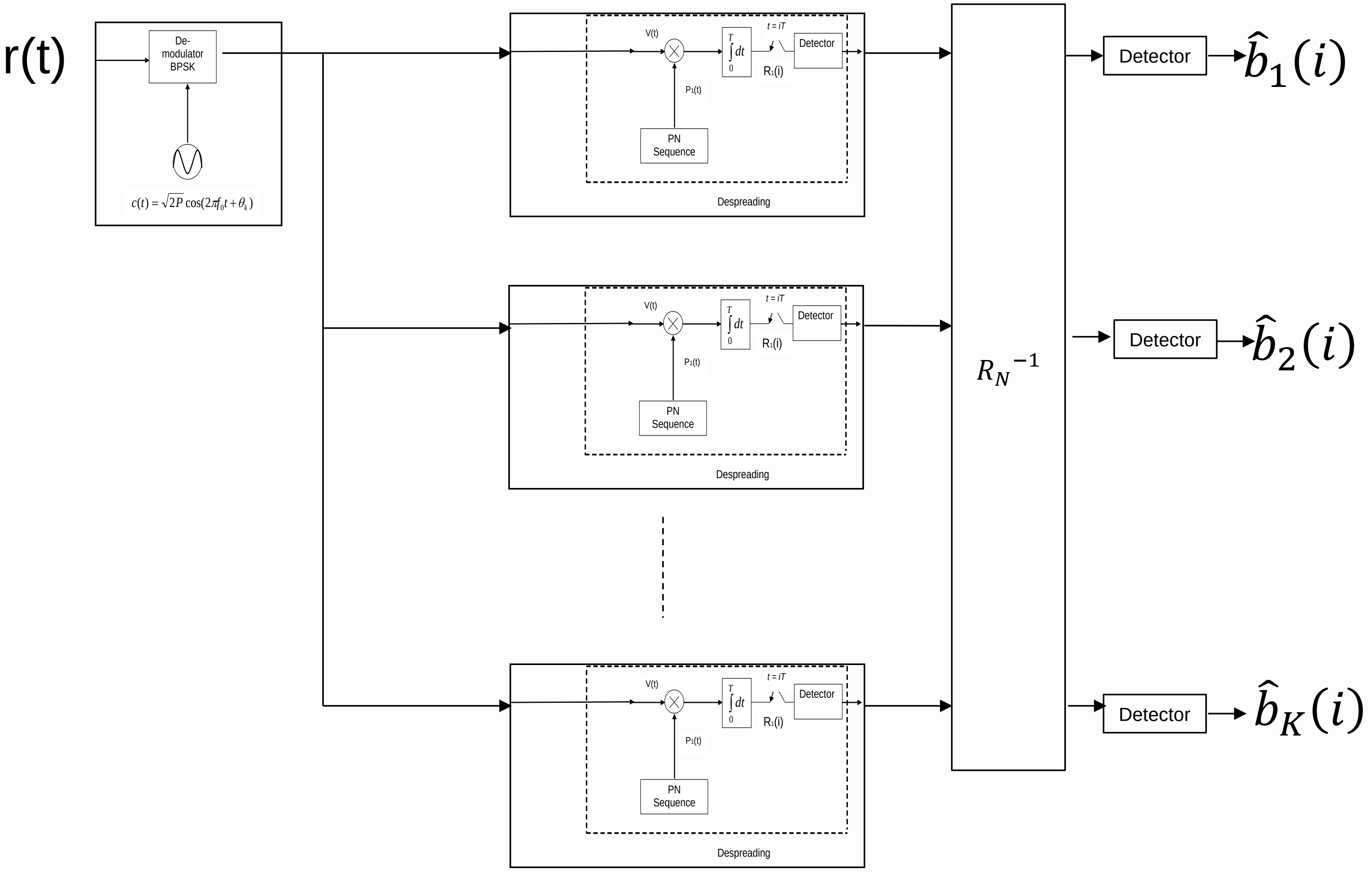
$$\boldsymbol{b}^{\mathbf{0}} = \boldsymbol{R}_{\boldsymbol{N}}^{-\mathbf{1}}\boldsymbol{R}_{\boldsymbol{N}}\boldsymbol{b} + \boldsymbol{R}_{\boldsymbol{N}}^{-\mathbf{1}}\boldsymbol{n} = \boldsymbol{b} + \boldsymbol{R}_{\boldsymbol{N}}^{-\mathbf{1}}\boldsymbol{n}$$

Moreover $\boldsymbol{E}\{\boldsymbol{b}^{\mathbf{0}}\} = \boldsymbol{E}\{\boldsymbol{b}\} + \boldsymbol{E}\{\boldsymbol{R}_{\boldsymbol{N}}^{-\mathbf{1}}\boldsymbol{n}\} = \boldsymbol{E}\{\boldsymbol{b}\}$

Showing thtat the mean of the solution is equal to the mean of the unknown variable to be estimated.

Therefore the DC estimator is an unbiased estimator of ***b****.*

Sub-optimal Receivers: Decorrelating Receiver



Sub-optimal Receivers: De-correlating Detector

Multiplication with inverse of matrix RN allows interference from all other users matched filter outputs to be subtracted, so keeping into account estimated intefering bits of other users. Decorrelating effect so arise. Let us consider as example a case with N=1, K=2, i.e. two users transmitting one bit each. Than matrix $\boldsymbol{R_N}$ reduces to

$R_N = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ and $R^{N^{-1}} = \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \frac{1}{1-\rho^2}$, with

$\rho_{12}(0) = \rho_{21}(0) = \rho = \int c_1(t-\tau)c_2(t)dt$. Then:

$$\boldsymbol{b^0} = \boldsymbol{R_N^{-1}r} = \begin{bmatrix} \frac{1}{1-\rho^2} & \frac{-\rho}{1-\rho^2} \\ \frac{-\rho}{1-\rho^2} & \frac{1}{1-\rho^2} \end{bmatrix} \begin{bmatrix} r_1(1) = \sqrt{\frac{P}{2}} T b_1 + n_1 \\ r_2(1) = \sqrt{\frac{P}{2}} T b_2 + n_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{P}{2}} T(b_1 - \rho b_2) + n_1{}' \\ \sqrt{\frac{P}{2}} T(b_2 - \rho b_1) + n_2{}' \end{bmatrix}$$

Where $n_1{}'$ and $n_2{}'$ are zero mean Gaussian noise with double variance wrt to n_1 (being the sum of two Gaussian iid variables). The decision is taken on a variable where the interference of other user is eliminated, despite the variance of error includes noise effects on all K detectors

Sub-optimal Receiver: Minimum Mean Square Error (MMSE) Detector

The solution is obtained by combining in a linear way the output of the matched filters bank. This corresponds to imposing the solution to be $\boldsymbol{b}^0 = \boldsymbol{A}\boldsymbol{r}$

Where matrix *A* is also unknown. **A** and $\boldsymbol{b}^{\mathbf{0}}$ have to be jointly estimated. A **mean square error (MSE)** functional is so defined that has to be minimized.

$$\boldsymbol{J}(\boldsymbol{b}.\boldsymbol{A}) = \boldsymbol{E}\left\{\left(\boldsymbol{b}-\boldsymbol{b}^{\mathbf{0}}\right)^{\boldsymbol{T}}\left(\boldsymbol{b}-\boldsymbol{b}^{\mathbf{0}}\right)\right\} = \boldsymbol{E}\{(\boldsymbol{b}-\boldsymbol{A}\boldsymbol{r})^{\boldsymbol{T}}(\boldsymbol{b}-\boldsymbol{A}\boldsymbol{r})\}$$

And the solution is searched for such that

$$\underset{\boldsymbol{A},\boldsymbol{b}}{\operatorname{argmin}}\, \boldsymbol{J}(\boldsymbol{b}.\boldsymbol{A})$$

It can be proven that the optimal value of A is:

$$\boldsymbol{A}^0 = (\boldsymbol{R}_N + \frac{1}{2} N_0 \boldsymbol{I})^{-1}$$

$$\Rightarrow \quad \boldsymbol{b}^0 = (\boldsymbol{R}_N + \frac{1}{2} N_0 \boldsymbol{I})^{-1} \boldsymbol{r}$$

Sub-optimal Receivers: MMSE Detector

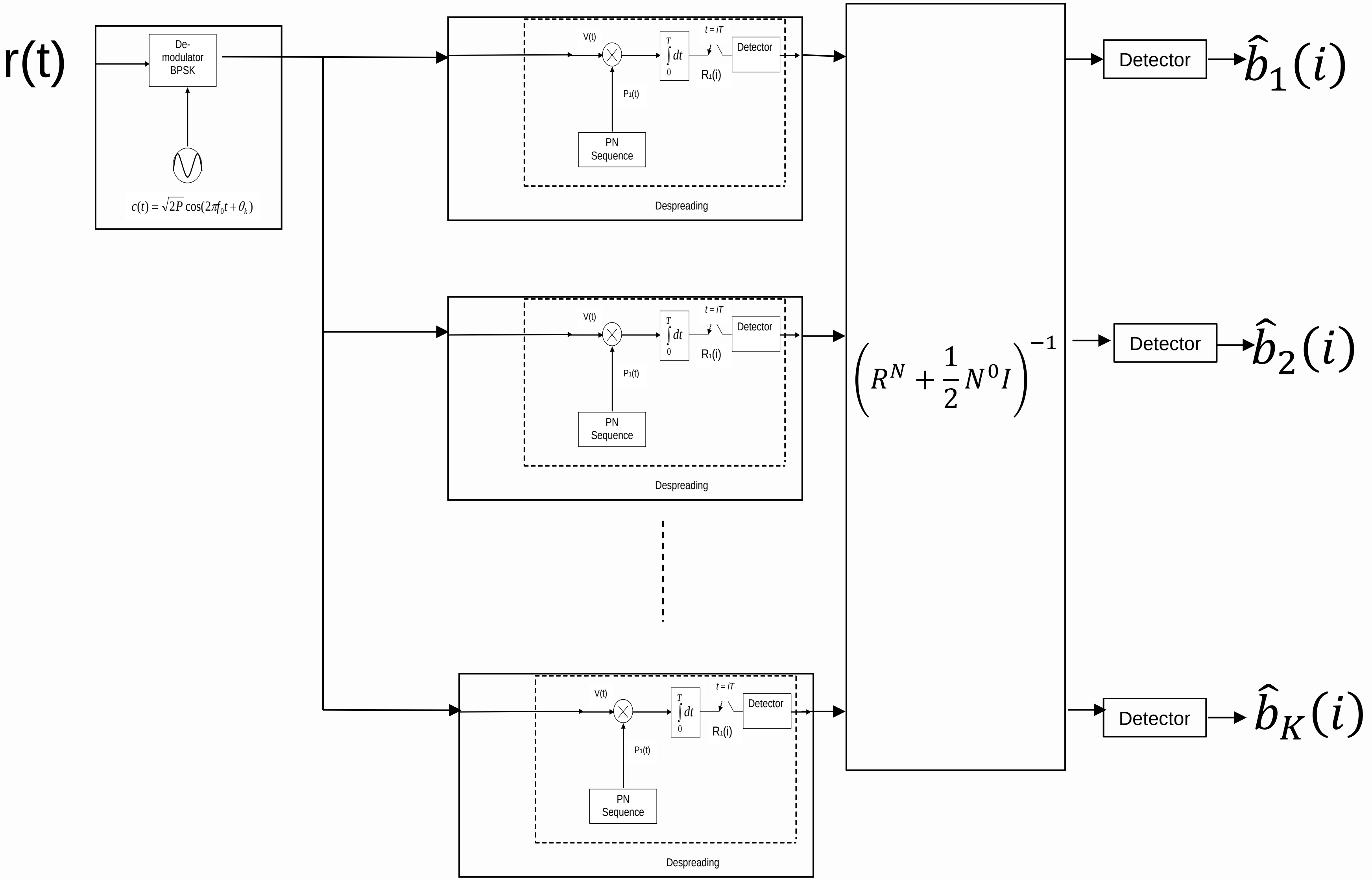
Recalling that the output of detector is: $\hat{\boldsymbol{b}} = sgn(\boldsymbol{b}^{0})$

- When Gaussian noise power $\frac{N^0}{2}$ is low compared to diagonal elements in $\boldsymbol{R_N}$, MMSE solution approximates ML solution of the de-correlating receiver, i.e. $\boldsymbol{A^0} = \boldsymbol{R_N}^{-1}$

- When Gaussian noise power $\frac{N^0}{2}$ is high with respect to diagonal elements matrix $\boldsymbol{A^0} = \frac{N^0}{2}\mathbf{I}$ is identical matrix (up to a scale factor) and a conventional receiver is substantially used. So detector substantially ignores MUI because channel Gaussian noise is dominant.

Minimum MSE detector provides a biased estimation of ***b***, thus there is a residual MUI.

Sub-optimal Receivers: MMSE Receiver



Sub-optimal Receiver: MMSE Detector

To obtain $\boldsymbol{b}^{\mathbf{0}}$, a linear system is to be computed: $(\boldsymbol{R}_N + \frac{1}{2} N_0 \boldsymbol{I})\boldsymbol{b} = \boldsymbol{r}$

An efficient solving method is the **square factorization(*)** of matrix:

$$\mathbf{R}_N + \frac{1}{2} N_0 \mathbf{I}$$

With this method $3NK^2$ multiplications are required to detect NK bits.

Computational load is $3K$ multiplications per bit and it is independent from block length N and increase linearly with K.