Continuity equation for incompressible fluid

Regardless of the flow assumptions, a statement of the conservation of mass is generally necessary. This is achieved through the mass continuity equation, given in its most general form as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

or, using the substantive derivative:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) = 0.$$

For incompressible fluid, density along the line of flow remains constant over time,

$$\frac{D\rho}{Dt} = 0.$$

therefore divergence of velocity is null all the time

$$\nabla \cdot \mathbf{u} = 0.$$

Stream function for incompressible 2D fluid

Taking the curl of the incompressible Navier–Stokes equation results in the elimination of pressure. This is especially easy to see if 2D Cartesian flow is assumed (like in the degenerate 3D case with $u_z = 0$ and no dependence of anything on z), where the equations reduce to:

$$\begin{aligned}
\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right) &= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho g_x \\
\rho\left(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y}\right) &= -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + \rho g_y.
\end{aligned}$$

Differentiating the first with respect to y, the second with respect to x and subtracting the resulting equations will eliminate pressure and any conservative force. For incompressible flow, defining the stream function ψ through

$$u_x = \frac{\partial \psi}{\partial y}; \quad u_y = -\frac{\partial \psi}{\partial x}$$

results in mass continuity being unconditionally satisfied (given the stream function is continuous), and then incompressible Newtonian 2D momentum and mass conservation condense into one equation:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2\psi\right) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2\psi\right) = \nu\nabla^4\psi$$

where ∇^4 is the 2D biharmonic operator and ν is the kinematic viscosity, $\nu = \frac{\mu}{\rho}$. We can also express this compactly using the Jacobian determinant:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right)+\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial(y,x)}=\nu\nabla^4\psi.$$

This single equation together with appropriate boundary conditions describes 2D fluid flow, taking only kinematic viscosity as a parameter. Note that the equation for creeping flow results when the left side is assumed zero.

In axisymmetric flow another stream function formulation, called the Stokes stream function, can be used to describe the velocity components of an incompressible flow with one scalar function.

The incompressible Navier–Stokes equation is a differential algebraic equation, having the inconvenient feature that there is no explicit mechanism for advancing the pressure in time. Consequently, much effort has been expended to eliminate the pressure from all or part of the computational process. The stream function formulation eliminates the pressure but only in two dimensions and at the expense of introducing higher derivatives and elimination of the velocity, which is the primary variable of interest.

Properties